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DECOMPOSITION INTO PARTIAL FRACTIONS.

By Eugene Randolph Smith.

In my former article on this subject in The Mathematics Teacher of June, 1914, I did not bring out one point, which, on further consideration, seems to me to be the key to this process. Dr. G. F. Metzler's excellent article in the issue of June, 1915, further emphasizes this need, for while he uses the point of which I speak in various ways, he does not bring the whole question to this focus.

Without elaboration, the fact is as follows: In the fraction $N/D\cdot F^k$ (where F is any prime factor of the denominator, D is the co-factor prime to F, and K is the positive integral power to which F appears) the numerators for the partial fractions of denominators F^k, F^{k-1}, $\cdots$ F are the successive coefficients in the quotient obtained from the detached coefficient function of F, $(R_1 + R_2 + \cdots R_k)/(r_1 + r_2 + \cdots r_k)$, R_1 to R_k being successive remainders obtained by dividing N by F, and r_1 to r_k being successive remainders obtained by dividing D by F.

It is evident that this reduces the method in partial fraction to

(1) Dividing the numerator and the co-factor in the denominator by the factor in question, repeating the division into each successive quotient until the number of divisions equals the number of partial fractions to be obtained.

(2) Dividing the function having the numerator remainders as coefficients by the function having the denominator remainders as coefficients.

(3) Simplifying the coefficients of the quotient (when necessary) and using them as the required numerators.

Illustrations.

(For the convenience of those who may wish to compare the methods, many of the examples used by Dr. Metzler are also used here.)

A. Involving the First Power of a Function of First Degree.

To obtain the numerator for $x+1$ from the fraction

$$x^2+1/[(x-1)^4(x^3+1)].$$

(No. 4 in Dr. Metzler's article.)

Divide* x^2+1 and $(x-1)^4(x^2-x+1)$ by $x+1$.

$$\begin{array}{r|l} \multicolumn{2}{c}{\textit{Numerator.}} \\ & 1+0+1 \\ -1 & 1-1\,\big|\overline{+2} \end{array} \qquad \begin{array}{c} \textit{Denominator.} \\ \\ (-2)^4\cdot 3 \end{array} \qquad = 48.$$

$$\therefore\ \frac{R_1}{r_1}=\frac{2}{48}=\frac{1}{24}$$

and the result is

$$\frac{1}{24(x+1)}.$$

B. Involving a Function of the First Degree Used to a Power Higher than the First.

To obtain the numerators for $(x-1)^4$, $(x-1)^3$, $\cdots$ $(x-1)$ in the same fraction.

Divide x^2+1 and x^3+1 by $x-1$ four times.

$$\begin{array}{r|l} \multicolumn{2}{c}{\textit{Numerator.}} \\ & 1+0+1 \\ & 1+1\,\big|\overline{+2}=R_1 \\ 1 & 1\,\big|+2=R_2 \\ & 1=R_3 \\ \multicolumn{2}{l}{0=\dot{R}_4} \end{array} \qquad\qquad \begin{array}{r|l} \multicolumn{2}{c}{\textit{Denominator.}} \\ & 1+0+0+1 \\ & 1+1+1\,\big|\overline{+2}=r_1 \\ 1 & 1+2\,\big|+3=r_2 \\ & 1\,\big|+3=r_3 \\ & 1=r_4 \end{array}$$

Take the quotient of these remainder functions

$$\begin{array}{r|r} & 2+2+1+0 \\ -1 & -1 \\ -3 & -3+\frac{3}{2} \\ -3 & -3+\frac{3}{2}+\frac{3}{4} \\ \hline 2 & 1-\frac{1}{2}-\frac{1}{4}+\frac{5}{8} \end{array}$$

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* The substitution of $x=-1$ from $x+1=0$ is an identical operation in such a case, and may be used if desired. It is simpler in the denominator of this example.

giving

$$\frac{1}{(x-1)^4}-\frac{1}{2(x-1)^3}-\frac{1}{4(x-1)^2}-\frac{5}{8(x-1)}.$$

In A and B the remainders were of necessity independent of x, and the results were therefore in simplest form. In the types to follow this is not necessarily true, for when the remainders $r_1 \cdots r_k$ are functions of x, the results obtained will involve x in the denominator and will therefore need further simplification. This possibility occurs when F is of degree higher than the first.

C. To Obtain the Numerator for $x^2 - x + 1$ in the Same Fraction.

Divide x^2+1 and $(x-1)^4(x+1)$ by x^2+x+1.

Numerator.

$$\begin{array}{r|lll} & 1 & +0 & +1 \\ -1 & & & -1 \\ 1 & & 1 & \\ \hline 1 & 1 & \multicolumn{1}{|l}{1} & 0 = R_1 \end{array}$$

Denominator.

$$\begin{array}{r|llllll} & 1 & -3 & +2 & +2 & -3 & +1 \\ -1 & & & -1 & 2 & 1 & -3 \\ 1 & & 1 & -2 & -1 & +3 & \\ \hline 1 & 1 & -2 & -1 & +3 & \multicolumn{1}{|l}{+1} & -2 = r_1 \end{array}$$

Then $R_1/r_1 = x/(x-2)$ which is not in simplest form.

The problem is therefore to transform $x/(x-2)$ into a fraction with numerical denominator, using the condition $x^2-x+1=0$.

Method I. By Auxiliary Factor.—Divide x^2-x+1 by $x-2$, obtaining a quotient $x+1$ and a remainder that is not to be used.

Multiply numerator and denominator of $x/(x-2)$ by $x+1$, giving $(x^2+x)/(x^2-x-2)$. Substitute $x^2=x-1$ from $x^2-x+1=0$, giving $(2x-1)/-3$ and the required partial fraction is $-(2x-1)/3(x^2-x+1)$.

The auxiliary factor can also be obtained by formula. For a fraction of denominator $mx+n$, when $ax^2+bx+c=0$, it is

$$\left[amx - \begin{vmatrix} m & n \\ a & b \end{vmatrix}\right].$$

This can be remembered by the fact that the first coefficient is the product of the two given first coefficients, while the second coefficient is the determinant of the first and second coefficients in the two given functions.

For example, to reduce $(3x-2)/(5x-7)$ to a fraction of numerical denominator when $2x^2-7x+1=0$.

The auxiliary factor is

$$\left\{2\cdot 5x-\begin{vmatrix}5 & -7\\ 2 & -7\end{vmatrix}\right\}=10x-21.$$

$$\frac{(3x-2)(10x-21)}{(5x-7)(10x-21)}=\frac{30x^2-83x+42}{50x^2-175x+147}.$$

Substituting $2x^2=7x-1$, this reduces to $(22x+27)/122$.

It would be quite possible to reduce such a fraction by substituting in it the value of x obtained from the quadratic, but that, in general, is a more difficult process.

Method II. By Formula.—The fraction $(kx+l)/(mx+n)$ when $ax^2+bx+c=0$, reduces to

$$\frac{a\begin{vmatrix}m & n\\ k & l\end{vmatrix}x-\begin{vmatrix}a & b & c\\ m & n & 0\\ 0 & k & l\end{vmatrix}}{-\begin{vmatrix}a & b & c\\ m & n & 0\\ 0 & m & n\end{vmatrix}}.$$

This arrangement of coefficients is easy to remember, and the determinant evaluates simply in terms of the last column.

Using the same example, $(3x-2)/(5x-7)$ when

$$2x^2-7x+1=0$$

the result is

$$\frac{2\begin{vmatrix}5 & -7\\ 3 & -2\end{vmatrix}x-\begin{vmatrix}2 & -7 & 1\\ 5 & -7 & 0\\ 0 & 3 & -2\end{vmatrix}}{-\begin{vmatrix}2 & -7 & 1\\ 5 & -7 & 0\\ 0 & 5 & -7\end{vmatrix}}=\frac{22x+27}{122}.$$

Method III. By Undetermined Coefficients.—Let

$$x/(x-2)=ax+b.$$

Then $x=ax^2+(b-2a)x-2b$, but since $x^2-x+1=0$ $ax^2=ax-a$, giving $(b-a)x-(a+2b)=x$.

Therefore $b-a=1$

$$a+2b=0 \quad \text{and} \quad b=1/3, \quad a=-2/3$$

giving, $(-2x+1)/3$.

Method IV. By Partial Fractions.—In $x/[(x-2)(x^2-x+1)]$, substituting $x=2$, the numerator for $x-2$ would be $\frac{2}{3}$; therefore subtracting $2/[3(x-2)]$ from this fraction, we obtain

$$\frac{3x-2(x^2-x+1)}{3(x-2)(x^2-x+1)}=\frac{-2x^2+5x-2}{3(x-2)(x^2-x+1)}=\frac{-2x+1}{3(x^2-x+1)}.$$

D. The only case not yet considered is that of an unfactorable denominator that occurs to some power, as in $N/[D(ax^2+bx+c)^k]$. The method is again that of obtaining k remainders by dividing N and D by ax^2+bx+c in successive divisions, and taking the quotient of the functions obtained to k terms. The coefficients evaluated for $ax^2+bx+c=0$ are the respective numerators. In $(x^2+2)/[(x-2)^2(x^2+1)^4(x^2-x+1)^2]$ (No. 5 in Dr. Metzler's article), to find the numerators for $(x^2-x+1)^2$, etc., and for $(x^2+1)^4$, etc.

Using x^2-x+1 as a divisor,

Numerator.

$$\begin{array}{r|l} & 1+0+2 \\ -1 & \phantom{1+0+{}}-1 \\ 1 & \phantom{1+{}}1 \\ \hline 1 & \underline{1}\,\rfloor\ \ 1+1=R_1 \\ & 1=R_2 \end{array}$$

Denominator.

$(x-2)^2(x^2+1)^4 \div (x^2-x+1)$ gives

$$r_1=-3, \quad r_2=14x-6.$$

The solution of the detached coefficient identity

$$\frac{(x+1)+(1)}{-3+(14x-6)}=(ax+b)+(cx+d)$$

will give the required numerators.

$$\begin{array}{r|l} & (1+1)+(1) \\ -14+6 & \dfrac{22-20}{3}+\dfrac{{}^*14}{3} \\ \hline -3 & -\dfrac{1+1}{3}-\dfrac{22-17}{9} \end{array}$$

giving

$$-\frac{x+1}{3(x^2-x+1)^2}-\frac{22x-17}{9(x^2-x+1)}$$

as the required fractions. In this case since r_1 is independent of x, the results are in simplest form without further change.

Using x^2+1 as a divisor,

$R_1=1,\ \ R_2=1,\ \ R_3=0,\ \ R_4=0,$

$r_1=4x-3,\ \ r_2=-10x-6,\ \ r_3=-6x+12,\ \ r_4=1.$

Then the required numerators can be obtained from

$$\frac{(1)-(1)}{(4x-3)+(-10x-6)+(-6x+12)+1}$$
$$=(ax+b)+(cx+d)+(ex+f)+(gx+h).$$

It will be observed in the following division that besides the terms of the divisor there is used another term, -4. This term and its products are marked. This is the first term of the divisor $4x-3$ used with sign changed, to correct the error which would otherwise be introduced by the following fact. As can be seen by clearing of fractions in the preceding identity, each multiplication of a term in the right hand member by $4x-3$ gives a second degree product, and therefore gives a multiple of x^2+1

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* The product $-\left(\frac{1+1}{3}\right)(-14+6)$ contains $\frac{14(x^2-x+1)}{3}+\frac{22x-20}{3}$

The $14(x^2-x+1)/3$ belongs in the next column as it is the next higher power of the factor x^2-x+1, in terms of which the identity is being considered. If three numerators were to be obtained, this term would need to be added in with the third column. This will be more evident in the process of finding the numerator for the powers of x^2+1, where the numbers carried forward in this way will also be indicated by the sign * and will be placed directly following the product from which they are obtained.

which should therefore be transferred to the next column. The starred terms correct the similar error for all terms of the divisor except the first. Note that only the x coefficient of the preceding term of the quotient should be multiplied by this number.

	$(0+1)$	$+(0+1)$	$+(0+0)$	$+(0+0)$
$-4^{\odot}$ (with x coefficient only)		$\frac{16^{\odot}}{25}$	$\frac{72^{\odot}}{125}$	$-\frac{2272^{\odot}}{625}$
$0-1$				
$6-12$			$\frac{30+60}{25}$	$-\frac{*24}{25}\ \frac{-1350+5400}{625}$
$10+6$		$\frac{-54+22}{25}$	$-\frac{*40}{25}\ \frac{-918-306}{125}$	$\frac{*180}{125}\ \frac{-1932-3204}{625}$
$4-3$	$-\frac{4+3}{25}$	$\frac{(54-63)(4+3)}{625}$	$\frac{(768+134)(4+3)}{3125}$	$\frac{(3182+7181)(4+3)}{25^{3}}$

or, inverted,*

$$-\frac{4+3}{25}$$

using $x^2+1=0$

$$-\frac{4+3}{25}-\frac{18x+81}{*125}\ \frac{568x-534}{625}\ \frac{7654x+1763}{3125}$$

giving the fractions

$$-\frac{(4x+3)}{25(x^2+1)^4}-\frac{18x+81}{125(x^2+1)^3}+\frac{568x-534}{625(x^2+1)^2}+\frac{7654x+1763}{3125(x^2+1)}$$

as a result.

While there may be some doubt as to whether this is the shortest possible process in all cases, numerous trials have seemed to show that it is the quickest method to apply in the usual examples. At any rate it has the advantage of being general in its application. An important special case is that when $r_1=1$, r_2 . . . being zero, then R_1, R_2, etc., are the required results.†

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† Since to divide by $4x-3$ is to multiply by $1/(4x-3)$ or $-(4x+3)/25$ (using $x^2+1=0$), it is simpler to use this form throughout, thus saving the duplication of this process on each denominator.

Special Case Simplification.

Often in questions having two factors in the denominator, and sometimes in other cases, it is easier to subtract the simpler partial fractions from the given fraction, and then to obtain the other numerators by the easy method of perfect powers.†

In

$$\frac{x^2 - x + 1}{(x^2 - x + 2)^2(x^2 + x + 3)^3}$$

(No. 6 in Dr. Metzler's article).

Find the fractions

$$\frac{29 + 10x}{1331(x^2 - x + 2)^2} + \frac{22x - 60}{1331(x^2 - x + 2)}$$

by the usual method.

Subtracting these fractions from the given fraction leaves a numerator $1/1331(-22x^9 + 16x^8 - 132x^7 + 315x^6 - 329x^5 + 1284x^4 - 755x^3 + 3743x^2 - 1952x + 3788)$:

Divide this twice by $x^2 - x + 2$ (evenly) then three times by $x^2 + x + 3$, leaving

$$R_1 = \frac{1}{1331}(242x + 242), \quad R_2 = \frac{1}{1331}(44x - 187),$$

$$R_3 = \frac{1}{1331}(-22x + 16).$$

The results

$$\frac{242x + 242}{1331(x^2 + x + 3)^3} - \frac{44x + 187}{1331(x^2 + x + 3)^2} - \frac{22x - 16}{1331(x^2 + x + 3)}$$

are obtained.

The Use of Determinants.

An interesting variation in the use of remainders to solve partial fractions, is the expression of the required results in determinants whose constituents are the remainders.

The numerators for successive descending powers of a first degree factor of the denominator, if $R_1, R_2, \ldots, R_n$ are the successive remainders obtained by dividing the numerator by that factor, and $r_1, r_2, \ldots, r_n$ are similar remainders from the denominator (exclusive of the factor in question), are

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† Both of these facts are used in Chap. xviv. of "A New Algebra," by Barnard and Childs.

$$\frac{R_1}{r_1}; \quad -\frac{\begin{vmatrix} R_1 & R_2 \\ r_1 & r_2 \end{vmatrix}}{r_1^2}; \quad \frac{\begin{vmatrix} R_1 & R_2 & R_3 \\ r_1 & r_2 & r_3 \\ 0 & r_1 & r_2 \end{vmatrix}}{r_1^3};$$

and, for the nth numerator,

$$\frac{(-1)^{n+1}}{r_1^n}\begin{vmatrix} R_1 & R_2 & & \cdots & R_n \\ r_1 & r_2 & & \cdots & r_n \\ 0 & r_1 & & \cdots & r_{n-1} \\ 0 & 0 & r_1 & \cdots & r_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & r_2 \end{vmatrix}.$$

The evaluation of these determinants is very simple, if it is done by multiplying the elements of the last column by their minors. This reduces each new determinant to the ones of lower order already evaluated.

$$\frac{9x^5 - 62x^4 + 171x^3 - 257x^2 + 239x - 90}{(x-2)^4(x^2+x+2)}.$$

Dividing by $x - 2$,

$$\begin{array}{|llll} R_1 = 24, & R_2 = -1, & R_3 = 1, & R_4 = 35 \\ r_1 = 8, & r_2 = 5, & r_3 = 1, & r_4 = 0 \end{array}$$

$$\frac{R_1}{r_1} = 3; \quad -\frac{\begin{vmatrix} R_1 & R_2 \\ r_1 & r_2 \end{vmatrix}}{r_1^2} = -\frac{\begin{vmatrix} 24 & -1 \\ 8 & 5 \end{vmatrix}}{8} = -2;$$

$$\frac{\begin{vmatrix} 24 & -1 & 1 \\ 8 & 5 & 1 \\ 0 & 8 & 5 \end{vmatrix}}{8^3} = \frac{5\begin{vmatrix} 24 & -1 \\ 8 & 5 \end{vmatrix} - 1\begin{vmatrix} 24 & -1 \\ 0 & 8 \end{vmatrix} + 1\begin{vmatrix} 8 & 5 \\ 0 & 8 \end{vmatrix}}{8^3}$$

$$= \frac{5 \cdot 128 - 1 \cdot 24 \cdot 8 + 1 \cdot 8 \cdot 8}{8^3} = \frac{10 - 3 + 1}{8} = 1.$$

Similarly

$$-\frac{\begin{vmatrix} 24 & -1 & 1 & 35 \\ 8 & 5 & 1 & 0 \\ 0 & 8 & 5 & 1 \\ 0 & 0 & 8 & 5 \end{vmatrix}}{8^4}$$

evaluates in terms of the previously used determinants, giving 4.

The solution is therefore

$$\frac{3}{(x-2)^4} - \frac{2}{(x-2)^3} + \frac{1}{(x-2)^2} + \frac{4}{(x-2)}.$$

When powers of second degree prime factors are used, the coefficients of the two term remainders are used, and the primed remainders gotten by carrying the division one term further in the denominator are also used.

If the remainders in the numerator are

$$S_1x + R_1; \quad S_2x + R_2; \quad S_3x + R_3, \text{ etc.,}$$

and in the denominator are

$$s_1x + r_1; \quad s_2x + r_2; \quad s_3x + r_3, \text{ etc.,}$$

with the remainders for the denominator co-factor times x

$$s_1'x + r_1'; \quad s_2'x + r_2', \text{ etc.,}$$

the successive numerators are, when

$$\Delta = \begin{vmatrix} s_1 & r_1 \\ s_1' & r_1' \end{vmatrix},$$

$$\frac{-x\begin{vmatrix} S_1 & R_1 \\ s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 \\ s_1' & r_1' \end{vmatrix}}{\Delta};$$

$$\frac{-x\begin{vmatrix} S_1 & R_2 & S_2 & R_2 \\ s_1 & r_1 & s_2 & r_2 \\ s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 & S_2 & R_2 \\ s_1 & r_1 & s_2 & r_2 \\ s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & s_1' & r_1' \end{vmatrix}}{\Delta^2};$$

$$\frac{-x\begin{vmatrix} S_1 & R_1 & S_2 & R_2 & S_3 & R_3 \\ s_1 & r_1 & s_2 & r_2 & s_3 & r_3 \\ s_1' & r_1' & s_2' & r_2' & s_3' & r_3' \\ 0 & 0 & s_1 & r_1 & s_2 & r_2 \\ 0 & 0 & s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & 0 & 0 & s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 & S_2 & R_2 & S_3 & R_3 \\ s_1 & s_1 & s_2 & r_2 & s_3 & r_3 \\ s_1' & r_1' & s_2' & r_2' & s_3' & r_3' \\ 0 & 0 & s_1 & r_1 & s_2 & r_2 \\ 0 & 0 & s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & 0 & 0 & s_1' & r_1' \end{vmatrix}}{\Delta^3}\ ;\ \text{etc.}$$

It will be seen that in this case the determinants should be evaluated in terms of the minors of the second order in the last two columns.

Using
$$\frac{x^2+2}{(x-2)^2(x^2+1)^4(x-x+1)^2}$$

(No. 5 in Dr. Metzler's article).

To find the numerators for successive powers of x^2+1, use x^2+1 as a divisor. The numerator remainders are seen by inspection to be 1, 1, 0, 0.

In the denominator, taking $(x-2)^2(x^2-x+1)^2$ as the dividend the following remainders are obtained.

$$\begin{array}{r|l}
 & 1-6+15-22+21-12+4 \\
-1 & \quad\ -1+\ 6-14+16-7\ \ -4 \\
0 & \\
\hline
1 & 1-6+14-16+\ 7\ \underline{+\ 4-3} = s_1+r_1 \\
-1 & \quad\ -1+\ 6-13+10\ \underline{-3-4} = s_1'+r_1' \\
0 & \\
\hline
1 & 1-6+13\ \underline{-10-6} = s_2+r_2 \\
-1 & \quad\ -1+\ 6\ \underline{-6+14} = s_2'+r_2' \\
0 & \\
\hline
1 & 1\ \underline{-6+12} = s_3+r_3 \\
 & \quad\ \underline{12-4} = s_3'+r_3'
\end{array}$$

$1=r_4$,

$1-6=s_4'+r_4'$.

Then the first numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -3 & 4 \end{vmatrix}}{\Delta},$$

when

$$\Delta = \begin{vmatrix} 4 & -3 \\ -3 & -4 \end{vmatrix} = -25,$$

giving

$$\frac{+4x+3}{-25} = -\frac{4x+3}{25}.$$

The second numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 & 0 & 1 \\ 4 & -3 & -10 & -6 \\ -3 & -4 & -6 & 14 \\ 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 \\ 4 & -3 & -10 & -6 \\ -3 & -4 & -6 & 14 \\ 0 & 0 & -3 & -4 \end{vmatrix}}{(-25)^2}$$

$$= \frac{-90x-405}{25} = -\frac{18x+81}{125}.$$

In evaluating these determinants but three products are used: in these products all the minors of the first two columns are known from the preceding evaluation, so the three second order minors of the last two columns are the only ones that need to be found. The third numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 \\ -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 \\ -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & -3 & -4 \end{vmatrix}}{(-25)^3}$$

$$= \frac{-14200x+13350}{(-25)^3} = \frac{568x-534}{625}.$$

In evaluating these determinants three second order minors are new, all other factors having been previously found.

The fourth numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 & 0 & 1 \\ -3 & -4 & -6 & 14 & 12 & -4 & 1 & -6 \\ 0 & 0 & 4 & -3 & -10 & -6 & -6 & 12 \\ 0 & 0 & -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 & 0 & 1 \\ -3 & -4 & -6 & 14 & 12 & -4 & 1 & -6 \\ 0 & 0 & 4 & -3 & -10 & -6 & -6 & 12 \\ 0 & 0 & -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & -4 \end{vmatrix}}{(-25)^4}$$

$$= \frac{7654x + 1763}{3125}$$

As in the other cases, there are but three second order minors to be found, and this rule holds for all such evaluations.

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